

SOME OSTROWSKI TYPE INEQUALITIES FOR DOUBLE INTEGRALS ON TIME SCALES

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ABSTRACT. The main objective of this paper is to study some Ostrowski and Trapezoid type inequalities for double integrals on Time Scales. Some other interesting inequalities are also given.

1. INTRODUCTION

In year 1988 the German mathematician in his Ph.D dissertation has initiated the study of time scales calculus which unifies the theory of both differential and difference calculus [9]. Dynamical equations and inequality's can be used studying various properties and model many phenomena in economics [5], biological systems [23] and various systems in neural network [11].

In [8] Bohner and Matthews have given the Ostrowski inequality and Montgomery identity on time scales. Some results on Ostrowski and Gruss inequality were obtained by N. Ahmad, W. Liu and others [2, 12, 22]. Recently in [4, 10, 13, 15, 24] authors have obtained some new Ostrowski type inequalities. Weighted Ostrowski and Trapezoid inequalities on time scales are obtained by W. Liu and others in [16, 17, 18]. In [21] M. Sarikya have studied some weighted Ostrowski and Chebsev type inequalities on time scales. Motivated by the results in the above paper we obtain some Ostrowski and Trapezoid type inequalities for double integrals on time scales.

In what follows the time scale \mathbb{T} is a nonempty closed subset of \mathbb{R} . Let $t \in \mathbb{T}$ the mapping $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ are defined as $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$ are called the forward and backward jump operators respectively.

We say that $f : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous provided f is continuous at each right-dense point of \mathbb{T} and has a finite left sided limit at each left dense point of \mathbb{T} . C_{rd} denotes the set of rd-continuous function defined on \mathbb{T} . Let \mathbb{T}_1 and \mathbb{T}_2 be two time scales with at least two points and consider the time scales intervals $\overline{\mathbb{T}}_1 = [x_0, \infty) \cap \mathbb{T}_1$ and

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$\overline{\mathbb{T}}_2 = [y_0, \infty) \cap \mathbb{T}_2$ for $x_0 \in \mathbb{T}_1$ and $y_0 \in \mathbb{T}_2$ and $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$. Let $\sigma_1, \rho_1, \Delta_1$ and $\sigma_2, \rho_2, \Delta_2$ denote the forward jump operators, backward jump operators and the delta differentiation operator respectively on \mathbb{T}_1 and \mathbb{T}_2 . Let $a < b$ be points in \mathbb{T}_1 , $c < d$ are point in \mathbb{T}_2 , $[a, b)$ is the half closed bounded interval in \mathbb{T}_1 , and $[c, d)$ is the half closed bounded interval in \mathbb{T}_2 .

We say that a real valued function f on $\mathbb{T}_1 \times \mathbb{T}_2$ at $(t_1, t_2) \in \overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2$ has a Δ_1 partial derivative $f^{\Delta_1}(t_1, t_2)$ with respect to t_1 if for each $\epsilon > 0$ there exists a neighborhood U_{t_1} of t_1 such that

$$|f(\sigma_1(t_1), t_2) - f(s, t_2) - f^{\Delta_1}(t_1, t_2)(\sigma_1(t_1) - s)| \leq \epsilon |\sigma_1(t_1) - s|,$$

for each $s \in U_{t_1}$, $t_2 \in \mathbb{T}_2$. We say that f on $\mathbb{T}_1 \times \mathbb{T}_2$ at $(t_1, t_2) \in \overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2$ has a Δ_2 partial derivative $f^{\Delta_2}(t_1, t_2)$ with respect to t_2 if for each $\eta > 0$ there exists a neighborhood U_{t_2} of t_2 such that

$$|f(t_1, \sigma_2(t_2)) - f(t_1, l) - f^{\Delta_2}(t_1, t_2)(\sigma_2(t_2) - l)| \leq \eta |\sigma_2(t_2) - l|,$$

for all $l \in U_{t_2}$, $t_1 \in \mathbb{T}_1$. The function f is called rd-continuous in t_2 if for every $\alpha_1 \in \mathbb{T}_1$, the function $f(\alpha_1, \cdot)$ is rd-continuous on \mathbb{T}_2 . The function f is called rd-continuous in t_1 if for every $\alpha_2 \in \mathbb{T}_2$ the function $f(\cdot, \alpha_2)$ is rd-continuous on \mathbb{T}_1 .

Let CC_{rd} denote the set of functions $f(t_1, t_2)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ where f is rd continuous in t_1 and t_2 . Let CC'_{rd} deontes the set of all functions CC_{rd} for which both the Δ_1 partial derivative and Δ_2 partial derivative exists and are in CC_{rd} .

The basic information on time scales and inequalities can be found in [1, 3, 6, 7].

2. Ostrowski Inequalities for double integrals on time scales

Now we give Ostrowski Inequalities for double integrals on time scales

Theorem 2.1. Let $f, g \in CC'_{rd}([a, b] \times [c, d], \mathbb{R})$ and $f^{\Delta_2 \Delta_1}(x, y)$, $g^{\Delta_2 \Delta_1}(x, y)$ exist rd-continuous on $[a, b] \times [c, d]$. Then

$$\begin{aligned} & \left| \int_a^b \int_c^d \{ [f(x, y)g(x, y) - \frac{1}{2} [P(f(x, y))g(x, y) \right. \\ & \quad \left. + P(g(x, y))f(x, y)] \} \Delta_2 y \Delta_1 x \right| \\ & \leq \frac{1}{8} (b-a)(d-c) \int_a^b \int_c^d [|g(x, y)| \|f^{\Delta_2 \Delta_1}\|_\infty \\ & \quad + f(x, y) \|g^{\Delta_2 \Delta_1}\|_\infty] \Delta_2 y \Delta_1 x. \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} P(f(x, y)) &= \frac{1}{2} [f(\sigma_1(s), y) + f(x, \sigma_2(t)) + f(x, d) + f(b, y)] \\ &\quad - \frac{1}{4} [f(\sigma_1(s), \sigma_2(t)) + f(\sigma_1(s), d) + f(b, \sigma_2(t)) + f(b, d)], \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} Q(f^{\Delta_2 \Delta_1}(x, y)) &= \int_{\sigma_1(s)}^b \int_{\sigma_2(t)}^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta - \int_{\sigma_1(s)}^b \int_y^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta \\ &\quad - \int_x^b \int_{\sigma_2(t)}^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta + \int_x^b \int_y^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta. \end{aligned} \quad (2.3)$$

Similarly $P(g(x, y))$ and $Q(f^{\Delta_2 \Delta_1}(x, y))$ are defined similar to (2.2) and (2.3).

Proof. From the hypotheses we have for $(x, y) \in [a, b] \times [c, d]$

$$\begin{aligned} &\int_{\sigma_1(s)}^b \int_{\sigma_2(t)}^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta \\ &= \int_{\sigma_1(s)}^x \left[\frac{\partial f(\eta, \tau)}{\Delta_1 \eta} \Big|_{\sigma_2(t)}^y \right] \Delta_1 \eta \\ &= \int_{\sigma_1(s)}^x \left[\frac{\partial f(\eta, y)}{\Delta_1 \eta} - \frac{\partial f(\eta, \sigma_2(t))}{\Delta_1 \eta} \right] \Delta_1 \eta \\ &= f(\eta, y) \Big|_{\sigma_1(s)}^x - f(\eta, \sigma_2(t)) \Big|_{\sigma_1(s)}^x \\ &= f(x, y) - f(\sigma_1(s), y) - f(x, \sigma_2(t)) + f(\sigma_1(s), \sigma_2(t)). \end{aligned} \quad (2.4)$$

Similarly we have

$$\begin{aligned} &\int_{\sigma_1(s)}^b \int_y^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta \\ &= -f(x, y) - f(\sigma_1(s), d) + f(x, d) + f(\sigma_1(s), y), \end{aligned} \quad (2.5)$$

$$\begin{aligned}
& \int_x^b \int_{\sigma_2(t)}^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta \\
& = -f(x, y) - f(b, \sigma_2(t)) + f(x, \sigma_2(t)) + f(b, \sigma_2(t)), \quad (2.6)
\end{aligned}$$

and

$$\int_x^b \int_y^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta = f(x, y) + f(b, d) - f(x, d) - f(b, y). \quad (2.7)$$

Adding above identities we have

$$\begin{aligned}
& 4f(x, y) - 2f(\sigma_1(s), y) - 2f(x, \sigma_2(t)) - 2f(x, d) - 2f(b, y) \\
& + f(\sigma_1(s), \sigma_2(t)) + f(\sigma_1(s), d) + f(b, \sigma_2(t)) + f(b, d) \\
& = \int_{\sigma_1(s)}^b \int_{\sigma_2(t)}^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta - \int_{\sigma_1(s)}^b \int_y^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta \\
& - \int_x^b \int_{\sigma_2(t)}^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta + \int_x^b \int_y^d \frac{\partial^2 f(\eta, \tau)}{\Delta_2 \tau \Delta_1 \eta} \Delta_2 \tau \Delta_1 \eta. \quad (2.8)
\end{aligned}$$

From (2.3), (2.4) and (2.8) we have

$$f(x, y) - P(f(x, y)) = \frac{1}{4} Q(f^{\Delta_2 \Delta_1}(x, y)), \quad (2.9)$$

for $x, y \in [a, b] \times [c, d]$.

Similarly for function g we have

$$g(x, y) - P(g(x, y)) = \frac{1}{4} Q(g^{\Delta_2 \Delta_1}(x, y)), \quad (2.10)$$

for $x, y \in [a, b] \times [c, d]$. Multiplying (2.9) and (2.10) by $g(x, y)$ and $f(x, y)$ and adding the resulting identities we get

$$\begin{aligned}
& 2f(x, y)g(x, y) - g(x, y)P(f(x, y)) - f(x, y)P(g(x, y)) \\
& = \frac{1}{4}g(x, y)Q(f^{\Delta_2 \Delta_1}(x, y)) + \frac{1}{4}f(x, y)Q(g^{\Delta_2 \Delta_1}(x, y)). \quad (2.11)
\end{aligned}$$

Integrating (2.11) over $[a, b] \times [c, d]$ we have

$$\int_a^b \int_c^d [f(x, y)g(x, y)]$$

$$\begin{aligned}
& -\frac{1}{2} [P(f(x, y))g(x, y) + P(g(x, y))f(x, y)] \Delta_2 y \Delta_1 x \\
& = \frac{1}{8} \int_a^b \int_c^d [Q(f^{\Delta_2 \Delta_1}(x, y))g(x, y) + Q(g^{\Delta_2 \Delta_1}(x, y))f(x, y)] \Delta_2 y \Delta_1 x.
\end{aligned} \tag{2.12}$$

From the properties of modulus we have

$$\begin{aligned}
|Q(f^{\Delta_2 \Delta_1}(x, y))| & \leq \int_a^b \int_c^d |f^{\Delta_2 \Delta_1}(t, s)| \Delta_2 s \Delta_1 t \\
& \leq \|f^{\Delta_2 \Delta_1}\|_\infty (b-a)(d-c),
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
|Q(g^{\Delta_2 \Delta_1}(x, y))| & \leq \int_a^b \int_c^d |g^{\Delta_2 \Delta_1}(t, s)| \Delta_2 s \Delta_1 t \\
& \leq \|g^{\Delta_2 \Delta_1}\|_\infty (b-a)(d-c).
\end{aligned} \tag{2.14}$$

From (2.12), (2.13) and (2.14) we have

$$\begin{aligned}
& \left| \int_a^b \int_c^d [f(x, y)g(x, y) - \frac{1}{2} [P(f(x, y))g(x, y) + P(g(x, y))f(x, y)] \Delta_2 y \Delta_1 x \right| \\
& \leq \frac{1}{8} \int_a^b \int_c^d [|g(x, y)| |Q(f^{\Delta_2 \Delta_1}(x, y))| \\
& \quad + |f(x, y)| |Q(g^{\Delta_2 \Delta_1}(x, y))|] \Delta_2 y \Delta_1 x \\
& \leq \frac{1}{8} \int_a^b \int_c^d [|g(x, y)| \int_a^b \int_c^d |f^{\Delta_2 \Delta_1}(t, s)| \Delta_2 s \Delta_1 t \\
& \quad + |f(x, y)| \int_a^b \int_c^d |g^{\Delta_2 \Delta_1}(t, s)| \Delta_2 s \Delta_1 t] \Delta_2 y \Delta_1 x
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{8} (b-a) (d-c) \int_a^b \int_c^d [|g(x, y)| \|f^{\Delta_2 \Delta_1}\|_\infty \\
&\quad + |f(x, y)| \|g^{\Delta_2 \Delta_1}\|_\infty] \Delta_2 y \Delta_1 x.
\end{aligned} \tag{2.15}$$

Which is required inequality.

. Now we give the continuous and Discrete equivalent version of above inequality where $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$

Corollary 2.1. (Continuous Case) If we put $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ we have

$$\begin{aligned}
&\left| \int_a^b \int_c^d [f(x, y) g(x, y) \right. \\
&\quad \left. - \frac{1}{2} [P(f(x, y)) g(x, y) + P(g(x, y)) f(x, y)] \right] dy dx \Big| \\
&\leq \frac{1}{8} (b-a) (d-c) \int_a^b \int_c^d [|g(x, y)| \|D_2 D_1 f\|_\infty \\
&\quad + |f(x, y)| \|D_2 D_1 g\|_\infty] dy dx.
\end{aligned}$$

$$\begin{aligned}
P(f(x, y)) &= \frac{1}{2} [f(x, c) + f(x, d) + f(a, y) + f(b, y)] \\
&\quad - \frac{1}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)],
\end{aligned}$$

$$\begin{aligned}
Q(D_2 D_1 f(x, y)) &= \int_a^x \int_c^y D_2 D_1 f(t, s) ds dt - \int_a^x \int_y^d D_2 D_1 f(t, s) ds dt \\
&\quad - \int_x^b \int_c^y D_2 D_1 f(t, s) ds dt + \int_x^b \int_y^d D_2 D_1 f(t, s) ds dt.
\end{aligned}$$

Similarly $P(g(x, y))$ and $Q(D_2 D_1 g(x, y))$, which is Ostrowski inequality for Double integral.

Corollary 2.2. (Discrete Case) If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ and $a = c = 0$, $b = k \in \mathbb{N}$ and $d = r \in \mathbb{N}$. Then

$$\left| \sum_{x=1}^k \sum_{y=1}^r \left\{ f(x, y) g(x, y) - \frac{1}{2} (g(x, y) P(f(x, y)) + f(x, y) P(g(x, y))) \right\} \right|$$

$$\leq \frac{1}{8}kr \sum_{x=1}^k \sum_{y=1}^r [|g(x, y)| \|\Delta_2 \Delta_1 f\|_\infty + |f(x, y)| \|\Delta_2 \Delta_1 g\|_\infty],$$

$$\begin{aligned} P(f(x, y)) &= \frac{1}{2} [f(x, 1) + f(x, r+1) + f(1, y) + f(k+1, y)] \\ &\quad - \frac{1}{4} [f(1, 1) + f(1, r+1) + f(k+1, 1) + f(k+1, r+1)], \end{aligned}$$

$$\begin{aligned} Q(\Delta_2 \Delta_1 f(x, y)) &= \sum_{s=1}^{x-1} \sum_{t=1}^{y-1} \Delta_2 \Delta_1 f(s, t) - \sum_{s=1}^{x-1} \sum_{t=1}^m \Delta_2 \Delta_1 f(s, t) \\ &\quad - \sum_{s=x}^k \sum_{t=1}^{y-1} \Delta_2 \Delta_1 f(s, t) + \sum_{s=x}^k \sum_{t=y}^r \Delta_2 \Delta_1 f(s, t). \end{aligned}$$

Which is Discrete Ostrowski Inequality.

Now we give some Ostrowski inequality for double integrals.

Theorem 2.2. Let $f, g, P(f(x, y)), P(g(x, y)), f^{\Delta_2 \Delta_1}, g^{\Delta_2 \Delta_1}$ be as in Theorem 2.1 then

$$\begin{aligned} &\left| \int_a^b \int_c^d \{f(x, y)g(x, y) - [P(f(x, y))g(x, y) + P(g(x, y))f(x, y) \right. \\ &\quad \left. - P(f(x, y))P(g(x, y))\} \Delta_2 y \Delta_1 x \right| \\ &\leq \frac{1}{16} \{(b-a)(d-c)\}^2 \|f^{\Delta_2 \Delta_1}\|_\infty \|g^{\Delta_2 \Delta_1}\|_\infty. \end{aligned} \quad (2.16)$$

for $[x, y] \in [a, b] \times [c, d]$.

Proof. From (2.9) and (2.10) we have

$$f(x, y) - P(f(x, y)) = \frac{1}{4} Q(f^{\Delta_2 \Delta_1}(x, y)), \quad (2.17)$$

and

$$g(x, y) - P(g(x, y)) = \frac{1}{4} Q(g^{\Delta_2 \Delta_1}(x, y)), \quad (2.18)$$

for $[x, y] \in [a, b] \times [c, d]$.

. Multiplying left hand side and right hand side of (2.17) and (2.18) we get

$$\begin{aligned} &f(x, y)g(x, y) - [f(x, y)P(g(x, y)) + g(x, y)P(f(x, y))] \\ &= \frac{1}{16} Q(f^{\Delta_2 \Delta_1}(x, y)) Q(g^{\Delta_2 \Delta_1}(x, y)). \end{aligned} \quad (2.19)$$

Integrating (2.19) over $[a, b] \times [c, d]$ and from the properties of modulus we have

$$\begin{aligned}
& \left| \int_a^b \int_c^d \{f(x, y) g(x, y) \right. \\
& \quad - [P(f(x, y)) g(x, y) + P(g(x, y)) f(x, y) \\
& \quad \left. - P(f(x, y)) P(g(x, y))\} \Delta_2 y \Delta_1 x \right| \\
& \leq \frac{1}{16} \int_a^b \int_c^d |Q(f^{\Delta_2 \Delta_1}(x, y))| |Q(g^{\Delta_2 \Delta_1}(x, y))| \Delta_2 y \Delta_1 x. \quad (2.20)
\end{aligned}$$

Now using (2.13) and (2.14) in (2.15) we get the required inequality (2.16).

. Now we give continuous and discrete version of the inequality (2.16) where $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ which is as follows

Corollary 2.3. (Continuous Case) If we put $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in above we get

$$\begin{aligned}
& \left| \int_a^b \int_c^d \{f(x, y) g(x, y) \right. \\
& \quad - [P(f(x, y)) g(x, y) + P(g(x, y)) f(x, y) \\
& \quad \left. - P(f(x, y)) P(g(x, y))\} dy dx \right| \\
& \leq \frac{1}{16} \int_a^b \int_c^d \|D_2 D_1 f\|_\infty \|D_2 D_1 g\|_\infty dy dx.
\end{aligned}$$

where $f, g, P, Q, D_2 D_1 f, D_2 D_1 g$ is as in Corollary 2.1. Which is Ostrowski type inequality for double integral.

Corollary 2.4. (Discrete Case) If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ and $a = c = 0, b = k \in \mathbb{N}$ and $d = r \in \mathbb{N}$. Then

$$\begin{aligned}
& \left| \sum_{x=1}^k \sum_{y=1}^r \{f(x, y) g(x, y) - [g(x, y) P(f(x, y)) + f(x, y) P(g(x, y)) \right. \\
& \quad \left. - P(f(x, y)) P(g(x, y))\} \right| \\
& \leq \frac{1}{16} (kr)^2 \int_a^b \int_c^d \|\Delta_2 \Delta_1 f\|_\infty \|\Delta_2 \Delta_1 g\|_\infty.
\end{aligned}$$

where P, Q are as in Corollary 2.2. Which is discrete Ostrowski type inequality.

3. TRAPEZOID TYPE INEQUALITY ON TIME SCALES

Now we give the dynamic Trapezoid type inequality on time scales.

Theorem 3.1. Let $f, f^{\Delta_2\Delta_1}$ be as in Theorem 2.1. Then

$$\begin{aligned}
& \left| \int_a^b \int_c^d f(t, s) \Delta_2 s \Delta_1 t - \frac{1}{2} \left[(d-c) \int_a^b f(t, \sigma_2(t)) + f(t, d) \right] \right. \\
& \quad \left. + (b-a) \int_c^d [f(\sigma_1(s), s) + f(b, s)] \Delta_2 s \right] \\
& \quad + \frac{1}{4} (b-a) (d-c) [f(\sigma_1(s), \sigma_2(t)) + f(\sigma_1(s), d) \\
& \quad + f(b, \sigma_2(t)) + f(b, d)] \\
& = \frac{1}{4} \int_a^b \int_c^d |f^{\Delta_2\Delta_1}(t, s)| \Delta_2 s \Delta_2 t. \tag{3.1}
\end{aligned}$$

Proof. From the proof of Theorem 2.1 we have

$$\begin{aligned}
f(x, y) &= \frac{1}{2} [f(\sigma_1(s), y) + f(x, \sigma_2(t)) + f(x, d) + f(b, y)] \\
& \quad + \frac{1}{4} [f(\sigma_1(s), \sigma_2(t)) + f(\sigma_1(s), d) + f(b, \sigma_2(t)) + f(b, d)] \\
& = \frac{1}{4} Q(f^{\Delta_2\Delta_1}(x, y)), \tag{3.2}
\end{aligned}$$

for $[x, y] \in [a, b] \times [c, d]$.

Integrating (3.2) over $[a, b] \times [c, d]$ we get

$$\begin{aligned}
& \int_a^b \int_c^d f(t, s) \Delta_2 s \Delta_1 t \\
& \quad - \frac{1}{2} \left[(d-c) \int_a^b [f(t, \sigma_2(t)) + f(t, d)] \Delta_1 t \right. \\
& \quad \left. + (b-a) \int_c^d [f(\sigma_1(s), s) + f(b, s)] \Delta_2 s \right] \\
& \quad + \frac{1}{4} (b-a) (d-c) [f(\sigma_1(s), \sigma_2(t)) + f(\sigma_1(s), d) \\
& \quad + f(b, \sigma_2(t)) + f(b, d)]
\end{aligned}$$

$$= \frac{1}{4} \int_a^b \int_c^d Q(f^{\Delta_2 \Delta_1}(t, s)) \Delta_2 s \Delta_1 t. \quad (3.3)$$

From the property of modulus and integrals we have

$$|Q(f^{\Delta_2 \Delta_1}(x, y))| \leq \int_a^b \int_c^d |f^{\Delta_2 \Delta_1}(t, s)| \Delta_2 s \Delta_1 t. \quad (3.4)$$

From (3.3) and (3.4) we have

$$\begin{aligned} & \left| \int_a^b \int_c^d f(t, s) \Delta_2 s \Delta_1 t \right. \\ & - \frac{1}{2} \left[(d - c) \int_a^b [f(t, \sigma_2(t)) + f(t, d)] \Delta_1 t \right. \\ & \left. + (b - a) \int_c^d [f(\sigma_1(s), s) + f(b, s)] \Delta_2 s \right] \\ & + \frac{1}{4} (b - a) (d - c) [f(\sigma_1(s), \sigma_2(t)) + f(\sigma_1(s), d) \\ & + f(b, \sigma_2(t)) + f(b, d)] \\ & \leq \frac{1}{4} \int_a^b \int_c^d |Q(f^{\Delta_2 \Delta_1}(t, s))| \Delta_2 s \Delta_1 t \\ & \leq \frac{1}{4} (b - a) (d - c) \int_a^b \int_c^d |f^{\Delta_2 \Delta_1}(t, s)| \Delta_2 s \Delta_1 t. \end{aligned} \quad (3.5)$$

which is required inequality. Now we give the Continuous and discrete version of Trapezoid inequality when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Corollary 3.1. (Continuous Case) If we put $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in above we get

$$\int_a^b \int_c^d f(t, s) ds dt - \frac{1}{2} \left[(d - c) \int_a^b [f(t, c) + f(t, d)] dt \right.$$

$$\begin{aligned}
& + (b-a) \int_c^d [f(a, s) + f(b, s)] ds \Bigg] \\
& + \frac{1}{4} (b-a) (d-c) [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \\
& \leq \frac{1}{4} (b-a) (d-c) \int_a^b \int_c^d |D_2 D_1 f(t, s)| dt ds.
\end{aligned}$$

which is Continuous Trapezoid type inequality

Corollary 3.2. (Discrete Case) If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ then we get

$$\begin{aligned}
& \left| \sum_{t=1}^k \sum_{s=1}^r f(t, s) - \frac{1}{2} \left[m \sum_{t=1}^k [f(t, 1) + f(t, m+1)] \right. \right. \\
& \quad \left. \left. + k \sum_{s=1}^r [f(1, s) + f(k+1, s)] \right] \right. \\
& \quad \left. + \frac{1}{4} km [f(1, 1) + f(1, r+1) + f(k+1, 1) + f(k+1, r+1)] \right|.
\end{aligned}$$

which is Discrete Trapezoid type inequality

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